Some Exact Analytical Solutions to the Inhomogeneous Higher-Order Nonlinear Schrödinger Equation Using Symbolic Computation

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In this paper, the generalized projective Riccati equation method is extended to investigate the inhomogeneous higher-order nonlinear Schrödinger (IHNLS) equation including not only the group velocity dispersion and self-phase-modulation, but also various higher-order effects, such as the third-order dispersion, self-steepening and self-frequency shift. With the help of symbolic computation, a broad class of analytical solutions of the IHNLS equation is presented, which include bright-like solitary wave solutions, dark-like solitary wave solutions, W-shaped solitary wave solutions, combined bright-like and dark-like solitary wave solutions, and dispersion-managed solitary wave solutions. From our results, many previously known results about the IHNLS equation can be recovered by means of some suitable selections of the arbitrary functions and arbitrary constants. Furthermore, from the soliton management concept, the main soliton-like character of the exact analytical solutions is discussed and simulated by computer under different parameters conditions.

Key words: Inhomogeneous Higher-Order Nonlinear Schrödinger Equation; Generalized Projective Riccati Equation Method; Solitary Wave Solutions; Symbolic Computation.

1. Introduction

As is known, the nonlinear Schrödinger equation (NLS) model is one of the most important and “universal” nonlinear models of modern science. In particular, NLS equation optical solitons are regarded as the natural data bits and as an important alternative for the next generation of ultrahigh speed optical telecommunication systems. The propagation of femtosecond light pulses in optical fibers can be described by the higher-order nonlinear Schrödinger (HNLS) equation including not only the group velocity dispersion (GVD) and the self-phase-modulation (SPM), but also various higher-order effects, such as the third-order dispersion (TOD), self-steepening, and self-frequency shift [1 – 3]. It has been extensively studied by many authors and some types of exact solitons or solitary wave solutions have been obtained [4 – 23]. It is worth noting that these investigations of optical solitons or solitary waves have been focused mainly on homogeneous fibers. However, in realistic fiber transmission lines, no fiber is homogeneous due to long distance communication and manufacturing problems. Recently, studies of the propagation of optical pulses in inhomogeneous fibers, which are described by the inhomogeneous nonlinear Schrödinger (INLS) equation, have attracted more interest. Many authors have investigated the INLS equations from different points of view and obtained some exact soliton solutions for special parameter relations [18 – 32]. Taking account of the higher-order effects, Papaioannou et al. first investigated the inhomogeneous higher-order nonlinear Schrödinger (IHNLS) equation, which describes femtosecond optical pulse propagation in inhomogeneous fibers, and derived exact bright and dark solitary wave solutions near the zero dispersion point [33].

The governing envelope wave equation for femtosecond optical pulse propagation in inhomogeneous fiber takes the form [33]

$$\begin{aligned} q_z = \mathrm{i}(\alpha_1(z)q_{tt} + \alpha_2(z)|q|^2 q) + \alpha_3(z)q_{ttt} \\ + \alpha_4(z)(|q|^2 q)_t + \alpha_5(z)q(|q|^2)_t + \Gamma(z)q, \end{aligned} \tag{1}$$

where $q(z,t)$ represents the complex envelope of the electrical field, z is the normalized propagation distance, t the normalized retarded time, and $\alpha_1(z)$, $\alpha_2(z)$, $\alpha_3(z)$, $\alpha_4(z)$, and $\alpha_5(z)$ are the distributed parameters, which are functions of the propagation distance z, related to GVD, SPM, TOD, self-steepening, and the delayed nonlinear response effect, respectively. $\Gamma(z)$ denotes the amplification or absorption coefficient. Study of (1) is of great interest due to its wide range of applications. Its use is not only restricted to optical pulse propagation in inhomogeneous fiber media, but also to the core of dispersion-managed solitons and combined managed solitons. In [34], exact multisoliton solutions of (1) are presented by employing the Darboux transformation based on the Lax pair. An exact dark soliton solution of (1) has been obtained in [35]. When $\Gamma(z) = 0$ in (1), two exact analytical solutions that describe modulation instability and soliton propagation on a continuous wave background are obtained by using the Darboux transformation [36]. By a direct assumption method, Yang et al. [37] presented three exact combined solitary wave solutions for (1), analyzed the features of the solutions and numerically discussed the stabilities of these solutions under slight violations of the parameter conditions and finite initial perturbations for some exact analytical solutions.

In this paper, we will extend the generalized projective Riccati equation method [38 – 40] to investigate the IHNLS equation (1). With the help of symbolic computation, five families of exact analytical solution of the IHNLS equation (1) are presented under some parametric conditions, which include bright solitary wave solutions, dark solitary wave solutions, combined bright-like and dark-like solitary wave solutions, and dispersion-managed solitary wave solutions. Furthermore, we analyze the features of the solutions. These results are useful for the design of fiber optic amplifiers and for studying the simultaneous propagation of bright and dark soliton-like pulses in femtosecond fiber laser systems or in optical communication links with distributed dispersion and nonlinearity management.

2. Exact Analytical Solutions of IHNLS Equation (1)

We now extend the projective Riccati equation method [38 – 40] to investigate the IHNLS equation (1). In order to obtain some exact solutions of the IHNLS equation (1), we first assume that the solutions of (1) are of the form

$$\begin{aligned} q(z,t) &= \Delta(z,t)\exp[\mathrm{i}\Theta(z,t)] \\ &= \Big[\mathrm{i}A_0(z) + \mathrm{i}A_1(z)\sigma(\xi) + B_1(z)\tau(\xi)\Big] \\ &\quad \cdot \exp\Big\{\mathrm{i}[k_0(z) + k_1(z)t + k_2(z)t^2]\Big\}, \end{aligned} \tag{2}$$

where

$$\xi = \eta(z)\,[t - \chi(z)], \tag{3}$$

and $A_0(z)$, $A_1(z)$, $B_1(z)$, $\eta(z)$, $\chi(z)$, $k_0(z)$, $k_1(z)$ and $k_2(z)$ are real functions of z to be determined. $\tau(\xi)$ and $\sigma(\xi)$ satisfy the following ordinary differential equations (ODEs) and a constraint condition:

$$\begin{aligned} \frac{\mathrm{d}\sigma(\xi)}{\mathrm{d}\xi} &= -\sigma(\xi)\tau(\xi), \\ \frac{\mathrm{d}\tau(\xi)}{\mathrm{d}\xi} &= 1 - \mu\sigma(\xi) - \tau^2(\xi), \end{aligned} \tag{4}$$

$$\tau^2(\xi) = 1 - 2\mu\sigma(\xi) + (\mu^2 - 1)\sigma^2(\xi), \tag{5}$$

where μ is a constant.

Substituting (2) – (5) into the IHNLS equation (1), removing the exponential term, collecting the coefficients of the monomials in $\tau(\xi)$, $\sigma(\xi)$ and t of the resulting system, then separating each coefficient to the real and imaginary parts and setting each part to zero, we obtain an over-determined system of ODEs with respect to the differentiable functions $\alpha_1(z)$, $\alpha_2(z)$, $\alpha_3(z)$, $\alpha_4(z)$, $\alpha_5(z)$, $\Gamma(z)$, $A_0(z)$, $A_1(z)$, $B_1(z)$, $\chi(z)$, $\eta(z)$, $k_0(z)$, $k_1(z)$ and $k_2(z)$. Because the ODE system includes 32 equations, for simplification, we omit them in the paper.

Solving the over-determined ODE system with the symbolic computation system *Maple*, we obtain the explicit expressions for $\alpha_1(z)$, $\alpha_2(z)$, $\alpha_3(z)$, $\alpha_4(z)$, $\alpha_5(z)$, $\Gamma(z)$, $A_0(z)$, $A_1(z)$, $B_1(z)$, $\chi(z)$, $\eta(z)$, $k_0(z)$, $k_1(z)$ and $k_2(z)$ or the constraints among them. For simplification, the solutions of the ODE system are also omitted in this paper.

We know that (4) and (5) have the solutions:

$$\sigma(\xi) = \frac{1}{\mu + \cosh\xi}, \quad \tau(\xi) = \frac{\sinh\xi}{\mu + \cosh\xi}. \tag{6}$$

Therefore, from (2), (3), (6) and the solutions of the over-determined ODE system, we can derive the following five families of exact analytical solutions of the IHNLS equation (1).

Remark 1. 1.) In [39], we investigated the inhomogeneous nonlinear Schrödinger (INLS) equation

$$\begin{aligned} &\mathrm{i}\frac{\partial u}{\partial z} + \frac{\alpha(z)}{2}\frac{\partial^2 u}{\partial t^2} + \beta(z)|u|^2 u \\ &+ M(z)t^2 u + \mathrm{i}F(z)u = 0, \end{aligned} \tag{7}$$

and assume that the solutions of the INLS equation are of the form

$$\begin{aligned} u(z,t) &= \Delta(z,t)\exp(\mathrm{i}\Theta(z,t)) \\ &= \Big[A_0(z)+A_1(z)\sigma(\xi)+B_1(z)\tau(\xi)\Big] \\ &\quad\cdot\exp\Big\{\mathrm{i}[k_0(z)+k_1(z)t+k_2(z)t^2]\Big\}, \\ \xi &= t\Omega(z)+\delta(z), \end{aligned} \tag{8}$$

where $\tau(\xi)$ and $\sigma(\xi)$ satisfy (4) and (5).

Thus from (2), (3) and (8), we can see that in this paper the function $\Delta(z,t)$ is extended to a complex function from the real function in [39]. Due to the introduction of the complex function $\Delta(z,t)$, the solutions of the IHNLS equation present more forms in a solution as we can see in the following paper.

2.) The form of solutions (2)−(5) assumed in the paper is more general than other previous forms presented by some authors. When $\mu=0$ and $k_2(z)=0$ in (2)−(5), the assumption in [37] can be recovered. When $\mu=k_1(z)=0$, $A_0(z)=A_1(z)=0$ (or $A_0(z)=B_1(z)=0$, the assumption in [26] can be recovered. At the same time, it is necessary to point out that when $\mu\neq 0$, the solutions of the ODE system are very complex, so only two cases are listed to illustrate the validity of the method.

3.) In [37], the sign of the term "iΓA" in (6) should be "−", so after (6) the sign of the terms with "Γ" should be corrected, or the sign of "$\Gamma(z)q$" in (1) should be changed into "−". Therefore comparing our results with the results obtained in [37], we should notice the sign of $\Gamma(z)$.

Family 1.

$$q_1 = A_1[\mathrm{i}\,\mathrm{sech}(\xi)+c_1\tanh(\xi)]\exp[\mathrm{i}(c_3t+k_0)], \tag{9}$$

where

$$\xi = c_2\left(t-\frac{(2c_1^2c_2^2-3c_1^2c_3^2+c_2^2+3c_3^2)\int\alpha_3\mathrm{d}z}{c_1^2-1}-c_5\right),$$

$$\alpha_1=-3c_3\alpha_3,\ \alpha_2=-c_3\alpha_4=2c_3\alpha_5, \tag{10}$$

$$\alpha_4=-3\frac{\alpha_3c_2^2}{A_1^2(c_1^2-1)},\Gamma=\frac{A_{1,z}}{A_1},\ k_0=2c_3^3\int\alpha_3\mathrm{d}z+c_4,$$

α_3 is an arbitrary function of z, and c_1, c_2, c_3, c_5 are arbitrary constants. [*Notice*: In the following, c_i ($i=0,1,\cdots,5$) denote arbitrary constants.]

Remark 2. From Family 1, we can derive

$$\Gamma=\frac{\alpha_{1,z}\alpha_2-\alpha_1\alpha_{2,z}}{2\alpha_1\alpha_2},\ A_1=c_0\exp\Big[\int\Gamma\mathrm{d}z\Big], \tag{11}$$

where c_0 is an integral constant, and the intensity of the solution (9) is given by

$$|q_1|^2=c_0^2\Big\{c_1^2+(1-c_1^2)\mathrm{sech}^2(\xi)\Big\}\exp\Big[2\int\Gamma\mathrm{d}z\Big]. \tag{12}$$

If further setting $c_0=\rho_0$, $c_0c_1=\lambda_0$, $c_3=k_c$, $c_2=\eta_c$, the solution (14) obtained in [37] can be recovered by our solution (9).

In this situation, the solution (9) describes bright-like or dark-like solitary waves (depending on the sign of $1-c_1^2$) with a variable platform $c_0^2c_1^2\exp[2\int\Gamma(z)\mathrm{d}z]$. We note that the time shift $\chi(z)$ and the group velocity $V(z)=\mathrm{d}\chi/\mathrm{d}z$ of the solitary wave are dependent on z, which leads to a change of the center position of the solitary wave along the propagation direction of the fiber, and means that we may design a fiber system to control the time shift and the velocity of the solitary wave. In order to understand the evolution of the solution (9), we consider a soliton management system similar to that of [26] under different parameters, where the system parameters are of the forms

$$\begin{aligned} &\alpha_3(z)=\beta\cos(\delta z),\ \alpha_1(z)=-3c_3\beta\cos(\delta z), \\ &\alpha_4=-3\frac{c_0^2c_2^2\beta}{c_1^2-1}\cos(\delta z)\exp(-2\gamma z), \\ &\alpha_2=-c_3\alpha_4,\ \alpha_5=-\frac{\alpha_4}{2},\ \Gamma(z)=\gamma, \end{aligned} \tag{13}$$

where β, δ, γ are constants.

In Fig. 1, $Q=|q(z,t)|^2$ denotes the intensity of solution. Figure 1a presents the evolution plots of a dark-like solitary wave solution (9) for the signs of $1-c_1^2=-3<0$ with $\Gamma=\gamma=-0.04<0$, which corresponds to a dispersion decreasing fiber. Figure 1b presents the interaction of two bright-like solitary wave solutions (9) with $1-c_1^2=0.84>0$. From Fig. 1, we can see that the intensity of the solitary wave decreases when $\Gamma=\gamma<0$, and the time shift and the group velocity of the solitary wave are changing while the solitary wave keeps its shape in propagation along the fiber. At the same time, it is necessary to point out that due to the periodic property of $\chi(z)=\frac{(2c_1^2c_2^2-3c_1^2c_3^2+c_2^2+3c_3^2)\int\alpha_3\mathrm{d}z}{c_1^2-1}=7.679999881\sin(z)$, the evolution of the intensity $|q_1|^2$

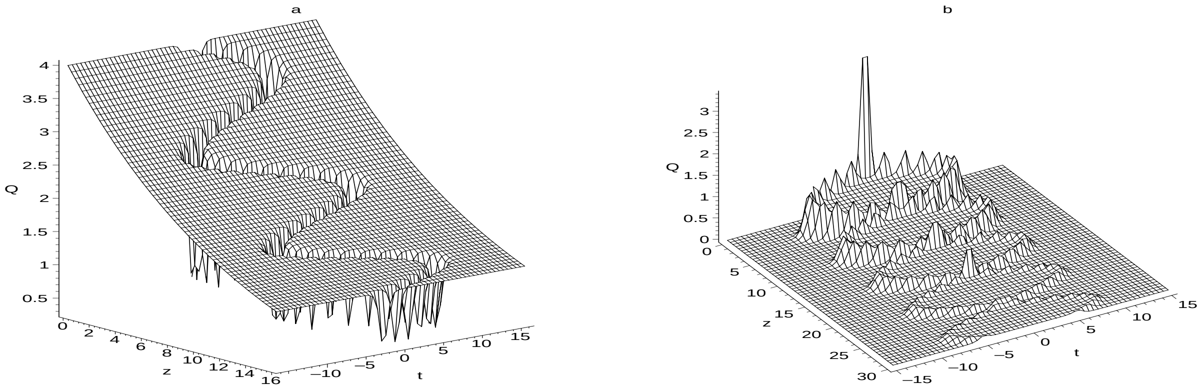


Fig. 1. The evolution plots of the solution (9) for a dark-like (a) and two bright-like (b) solitary wave interactions with parameters as follows: $\Gamma(z) = -0.04$, $c_0 = 1$, $c_2 = 1.6$, $c_3 = 0.0002$, $c_5 = 0$; (a) $\alpha_3(z) = \cos(z)$, $c_1 = 2$; and (b) $\alpha_3(z) = \cos(z/2)$, $c_1 = 0.4$.

demonstrates the periodic property. The cause of the typical period of the oscillating behavior for other solutions is similar.

Family 2.

$$q_{21} = A_0[\mathrm{i} + \mathrm{i}c_1\operatorname{sech}(\xi) \pm c_1\tanh(\xi)]\exp[\mathrm{i}(c_4t + k_0)], \tag{14}$$

and its intensity is given by

$$|q_{21}|^2 = A_0^2[1 + c_1^2 + 2c_1\operatorname{sech}(\xi)], \tag{15}$$

where

$$\begin{aligned}
\xi &= c_2t + (2c_1c_4 + c_2)\int A_0^2\alpha_4\mathrm{d}z - c_8,\\
\alpha_3 &= 0, \quad \alpha_5 = -\alpha_4, \quad \Gamma = \frac{A_{0,z}}{A_0},\\
\alpha_1 &= -\frac{c_1A_0^2\alpha_4}{c_2}, \quad \alpha_2 = \frac{\alpha_4(c_1c_2 - 2c_4)}{2},\\
k_0 &= \frac{(c_1^3c_2^2 + 2c_1c_4^2 + c_1c_2^2)\int A_0^2\alpha_4\mathrm{d}z}{2c_2} + c_6.
\end{aligned} \tag{16}$$

$$q_{22} = A_1\Big[\frac{\mathrm{i}}{\mu + \cosh(\xi)} \pm \frac{1}{\sqrt{1-\mu^2}}\frac{\sinh(\xi)}{\mu + \cosh(\xi)}\Big]\exp[\mathrm{i}(c_4t + k_0)], \tag{17}$$

and its intensity is given by

$$|q_{22}|^2 = A_1^2\frac{1 - \mu^2 + \sinh^2(\xi)}{(1-\mu^2)(\mu + \cosh^2(\xi))^2}, \tag{18}$$

where $0 < |\mu| < 1$ and

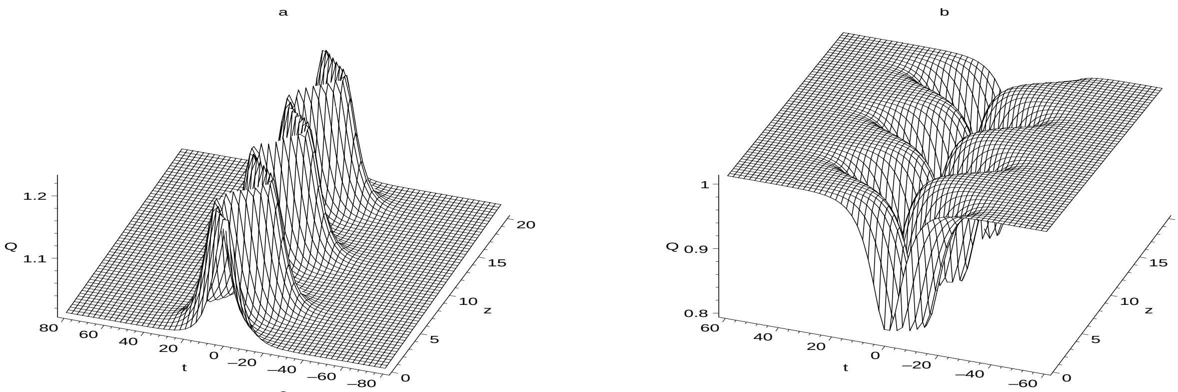
$$\begin{aligned}
\xi &= c_2t - \frac{c_4\mu\int A_1^2\alpha_4\mathrm{d}z}{\sqrt{1-\mu^2}(\mu^2-1)} + c_7,\\
\alpha_3 &= 0, \quad \alpha_5 = -\alpha_4, \quad A_1 = c_0\exp\Big[\int\Gamma(z)\mathrm{d}z\Big],\\
\alpha_1 &= \frac{\mu\alpha_4A_1^2}{c_2(\mu^2-1)\sqrt{1-\mu^2}},\\
\alpha_2 &= -c_4\alpha_4 + \frac{c_2\alpha_4}{2\mu\sqrt{1-\mu^2}},\\
k_0 &= -\frac{(c_2^2 - 2\mu^2c_4^2)\int A_1^2\alpha_4\mathrm{d}z}{c_2(\mu^2-1)\sqrt{1-\mu^2}} + c_6.
\end{aligned} \tag{19}$$

Remark 3. From (16), we can derive

$$\Gamma = \frac{\alpha_{1,z}\alpha_4 - \alpha_1\alpha_{4,z}}{2\alpha_1\alpha_4}, \; A_0 = c_0\exp\Big[\int\Gamma(z)\mathrm{d}z\Big], \tag{20}$$

where c_0 is an integration constant. If setting $c_0 = \beta_0$, $c_1c_0 = \lambda_0$, $c_2 = \eta_c$, $c_4 = k_c$, we can show that the solution (21) obtained in [37] can be recovered by (14) along with (16). To our knowledge, the solution (17) has not been reported earlier.

In this case, the solution (14) presents a bright-like or dark-like solitary wave depending on the sign of c_1. Figure 2a and Fig. 2b show the evolution plots of the solution (14) for different signs of c_1. This feature indicates that a bright and dark solitary wave may combine together under certain conditions and propagate simultaneously in an inhomogeneous fiber in combined form. Under $0 < \mu < 1$, the solution (17) presents a bright-like solitary wave.



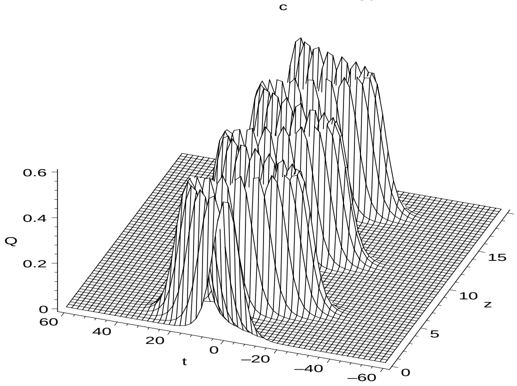


Fig. 2. The evolution plots of the solutions (14) and (17) with parameters as follows: $\alpha_4(z) = \exp(-0.0008z)\cos(z)$, $\Gamma(z) = 0.0004$, $c_0 = 1$, $c_2 = -0.2$, $c_4 = 10$, $c_7 = c_8 = 0$; (a) $c_1 = 0.1$; (b) $c_1 = -0.1$; and (c) $\mu = 0.3$.

Family 3.

$$q_3 = A_0 \left[\mathrm{i} + \mathrm{i}\frac{c_1}{\mu + \cosh(\xi)} \mp \frac{\Theta}{\Delta} \frac{\sinh(\xi)}{\mu + \cosh(\xi)} \right] \cdot \exp[\mathrm{i}(c_5 t + k_0)], \tag{21}$$

and its intensity is

$$|q_3|^2 = A_0^2 \left[(1 + \frac{c_1}{\mu + \cosh(\xi)})^2 + \frac{\Theta^2}{\Delta^2} \frac{\sinh^2(\xi)}{(\mu + \cosh(\xi))^2} \right], \tag{22}$$

where

$$\xi = c_3(t - c_8), \quad \alpha_1 = \alpha_3 = k_2 = 0,$$

$$\alpha_5 = -\frac{3}{2}\alpha_4, \quad A_0 = c_0 \exp\left[\int \Gamma \mathrm{d}z \right],$$

$$\alpha_2 = \pm(\frac{c_3 \Theta}{2c_1 \Omega} - c_5)\alpha_4,$$

$$k_0 = \frac{1}{2} \frac{[(\mu + c_1)^2 - 1](\mu + c_1)}{c_1 \Delta \Omega} \int \alpha_4 A_0^2 \mathrm{d}z + 2c_7, \tag{23}$$

and

$$\Theta = \sqrt{\Delta c_1 (2\mu^2 - 2 + 3c_1\mu + c_1{}^2)},$$

$$\Delta = -\mu + \mu^3 + c_1 + c_1 \mu^2,$$

$$\Omega = -1 + \mu^2 + c_1 \mu. \tag{24}$$

Remark 4. When setting $\mu = 0$ in solution (21), it is not difficult to derive that the solution (27) obtained in [37] can be recovered. Therefore the solution (21) has a more general form than the earlier reported one [12, 37].

As shown in Fig. 3, under different pulse parameters, the shapes of the intensity (22) take a bright-like solitary wave, dark-like solitary wave, W-shaped solitary wave and a combined bright and dark solitary wave, respectively. It is worth noting that, unlike the previous two cases, the solitary velocity does not change and there is only a changeless time shift in propagation due to the constant c_8. In particular, as shown in Fig. 3d, a bright solitary wave and a dark

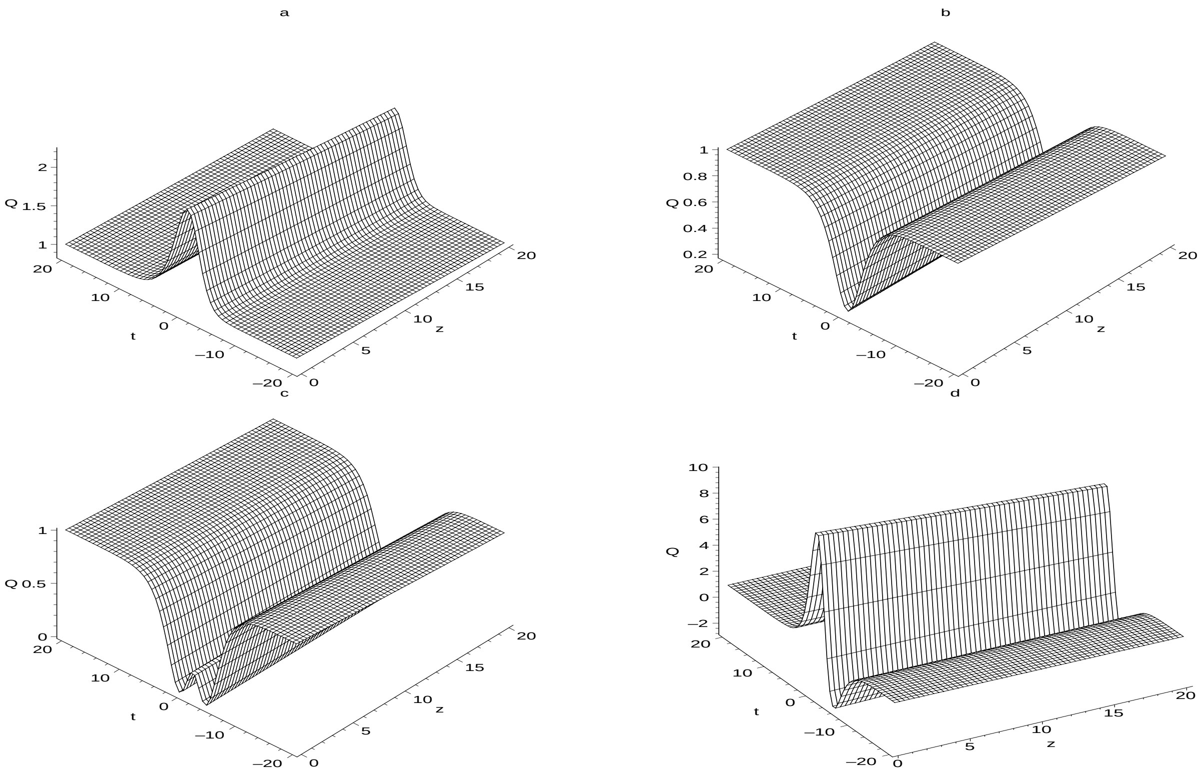


Fig. 3. The evolution plots of the solution (21) with parameters as follows: $\Gamma(z) = 0.0004$, $c_0 = 1$, $c_3 = 0.6$, $c_8 = -2$, $\mu = 1$; (a) $c_1 = 0.9$; (b) $c_1 = -0.9$; (c) $c_1 = -3$; and (d) $c_1 = -8$.

solitary wave are found to combine under some conditions and propagate simultaneously in an inhomogeneous fiber.

Family 4.

$$q_{41} = c_0 \exp\left[\int \Gamma \mathrm{d}z\right] \tanh(\xi) \exp(\mathrm{i}(c_2 t + k_0)),$$
$$|q_{41}|^2 = c_0^2 \exp\left[2\int \Gamma \mathrm{d}z\right] \tanh^2(\xi), \quad (25)$$

where

$$\xi = c_1\left[t - 2c_2 \int \alpha_1 \mathrm{d}z - (2c_1^2 + 3c_2^2) \int \alpha_3 \mathrm{d}z - c_4\right],$$

$$k_0 = -(c_2^2 + 2c_1^2) \int \alpha_1 \mathrm{d}z - (c_2^3 + 6c_2 c_1^2) \int \alpha_3 \mathrm{d}z + c_3,$$

$$\alpha_2 = -\frac{c_2 {B_1}^2 \alpha_4 + 2 {c_1}^2 \alpha_1 + 6 c_1^2 c_2 \alpha_3}{{B_1}^2},$$

$$\alpha_5 = -\frac{3}{2}\left(\alpha_4 + \frac{2c_1^2 \alpha_3}{B_1^2}\right),\ B_1 = c_0 \exp\left[\int \Gamma \mathrm{d}z\right]. \quad (26)$$

$$q_{42} = \mathrm{i}c_0 \exp\left[\int \Gamma \mathrm{d}z\right] \mathrm{sech}(\xi) \exp[\mathrm{i}(c_2 t + k_0)],$$
$$|q_{42}|^2 = c_0^2 \exp\left[2\int \Gamma \mathrm{d}z\right] \mathrm{sech}^2(\xi), \quad (27)$$

where

$$\xi = c_1\left(t - \int (3\alpha_3 c_2^2 - c_1^2 \alpha_3 + 2c_2 \alpha_1) \mathrm{d}z - c_4\right),$$

$$\alpha_2 = \frac{-\alpha_4 {A_1}^2 c_2 + 6 c_2 {c_1}^2 \alpha_3 + 2 {c_1}^2 \alpha_1}{{A_1}^2},$$

$$k_0 = \int (c_1^2 \alpha_1 - c_2^3 \alpha_3 - c_2^2 \alpha_1 + 3c_2 c_1^2 \alpha_3) \mathrm{d}z + c_3,$$

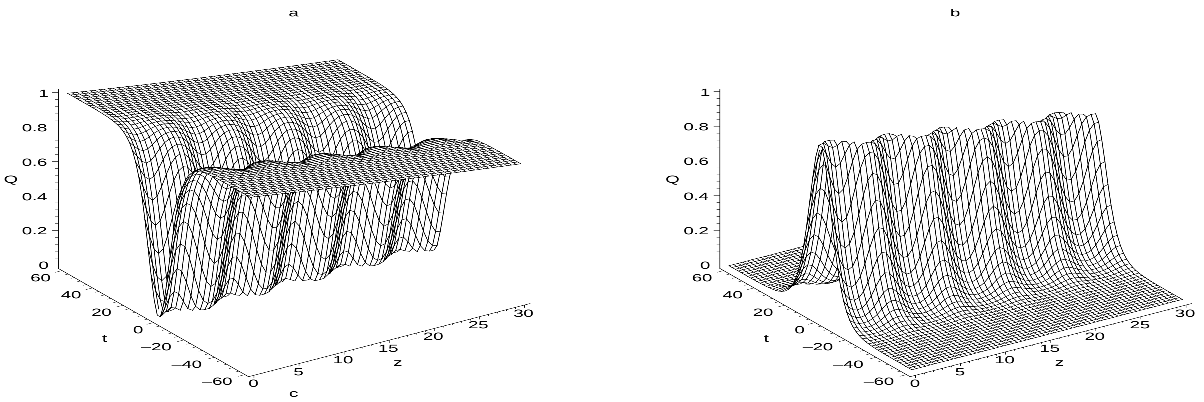
$$\alpha_5 = \frac{3}{2} \frac{-\alpha_4 {A_1}^2 + 2{c_1}^2 \alpha_3}{{A_1}^2}, A_1 = c_0 \exp\left[\int \Gamma \mathrm{d}z\right]. \quad (28)$$

$$q_{43} = c_0 \exp\left[\int \Gamma \mathrm{d}z\right] [\mathrm{i} + c_1 \tanh(\xi)] \exp\left[\mathrm{i}(c_3 t + k_0)\right],$$

$$|q_{43}|^2 = c_0^2 \exp\left[2\int \Gamma \mathrm{d}z\right] \left[1 + c_1^2 \tanh^2(\xi)\right], \quad (29)$$

where

$$\xi = c_2 t - \frac{c_2}{c_1^2}\left[(2c_1^2 c_2^2 + 3c_1^2 c_3^2 + 6c_1 c_2 c_3 + 6c_2^2) \int \alpha_3 \mathrm{d}z\right.$$



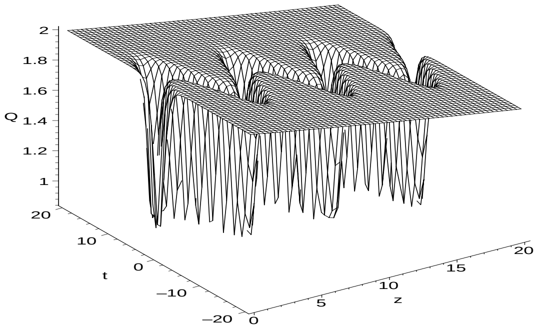


Fig. 4. The evolution plots of the solutions (25), (27) and (29) under the following parameters: $\Gamma(z) = -0.004$, $\alpha_1 = \alpha_3 = \cos(z)$, $c_0 = 1$, $c_2 = 0.7$, $c_3 = 0.01$, $c_4 = c_5 = 0$; (a, b) $c_1 = 0.1$; and (c) $c_1 = 1$.

$$
\begin{aligned}
&+ (2c_1^2c_3 + 2c_1c_2)\int \alpha_1 \mathrm{d}z\Big] - c_5,\\
&k_0 = -\frac{1}{c_1^3}\int\Big[(c_1{}^3c_3{}^3 + 6\,c_1{}^3c_2{}^2c_3 + 6\,c_1{}^2c_2{}^3\\
&+ 6\,c_1\,c_2{}^2c_3 + 6\,c_2{}^3)\alpha_3 + (c_1{}^3c_3{}^2 + 2\,c_1\,c_2{}^2\\
&+2\,c_2{}^2c_1{}^3)\alpha_1 + (c_2c_1{}^4(A_0)^2 + c_2c_1{}^2(A_0)^2)\alpha_4\Big]\mathrm{d}z + c_4,\\
&\alpha_5 = -\frac{3}{2}\,\frac{c_1{}^2(A_0)^2\alpha_4 + 2\,\alpha_3c_2{}^2}{c_1{}^2(A_0)^2},\\
&A_0 = c_0\exp\Big[\int \Gamma \mathrm{d}z\Big],\\
&\alpha_2 = -\frac{1}{A_0^2c_1^3}\Big[(6c_1c_2^2c_3 + 6c_2^3)\alpha_3\\
&\quad + (c_2c_1^2A_0^2 + c_1^3A_0^2c_3)\alpha_4 + 2c_1c_2^2\alpha_1\Big]. \quad (30)
\end{aligned}
$$

In the above results, $\alpha_1, \alpha_3, \alpha_4$ are all arbitrary functions of z.

Remark 5. When setting $\alpha_3 = \alpha_4 = \alpha_5 = 0$, it is not difficult to verify that the solutions (48) and (49) obtained in [22] can be recovered by our solutions (25) and (27). But to our knowledge, when $\alpha_3 \neq 0, \alpha_4 \neq 0$, and $\alpha_5 \neq 0$, the solutions in Family 4 have not been reported earlier.

Figure 4 presents the evolution plots of the solutions (25), (27) and (29) for the gain and/or loss distribution function $\Gamma(z) < 0$. In this case, the solutions (25) and (29) present a dark-like solitary wave and the solution (27) presents a bright-like solitary wave. As shown in Fig. 4, the height of the solitary waves (25), (27) and (29) decrease exponentially due to the exponentially decreasing nature of $c_0\exp(\gamma z)$.

Family 5.

$$
\begin{aligned}
&q_{51} = \mathrm{i}A_1\operatorname{sech}(\xi)\exp[\mathrm{i}(k_2(t^2 + c_2t) + k_{01})],\\
&|q_{51}|^2 = A_1^2\operatorname{sech}^2(\xi), \qquad (31)
\end{aligned}
$$

where

$$
\alpha_1 = -\frac{1}{4}\frac{k_2'}{k_2^2},\quad \alpha_2 = -\frac{1}{2}\frac{c_1^2k_2'}{A_1^2},
$$

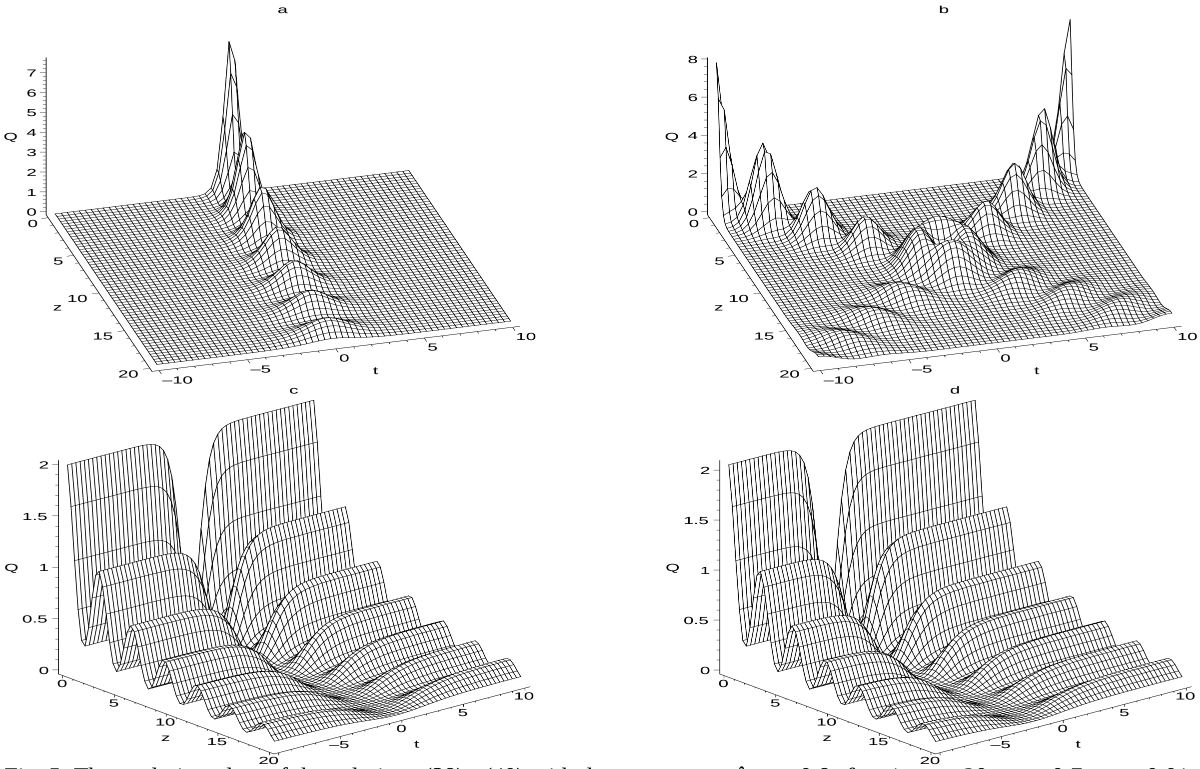


Fig. 5. The evolution plots of the solutions (38) – (40) with the parameters: $\lambda = -0.8$, $\theta = 1$, $c_0 = 20$, $c_2 = 0.7$, $c_3 = 0.01$, $c_4 = 0$; (a – c) $c_1 = 40$, $c_2 = 0.1$; (d) $c_1 = 6$, $c_2 = 20$.

$$\Gamma = \frac{-A_1 k_2' + 2A_1' k_2}{2A_1 k_2}, \quad \xi = c_1 k_2 (t + \frac{c_2}{2}) + c_4,$$

$$k_{01} = \frac{c_2^2 - c_1^2}{4} k_2 + c_3. \tag{32}$$

$$q_{52} = B_1 \tanh(\xi) \exp\left[\mathrm{i}(k_2(t^2 + c_2 t) + k_{02})\right],$$

$$|q_{52}|^2 = B_1^2 \tanh^2(\xi), \tag{33}$$

where

$$\alpha_1 = -\frac{1}{4}\frac{k_2'}{k_2^2}, \quad \alpha_2 = \frac{1}{2}\frac{c_1^2 k_2'}{B_1^2},$$

$$\Gamma = \frac{-B_1 k_2' + 2B_1' k_2}{2B_1 k_2}, \quad \xi = c_1 k_2 (t + \frac{c_2}{2}) + c_4,$$

$$k_{02} = \frac{c_2^2 + 2c_1^2}{4} k_2 + c_3. \tag{34}$$

$$q_{53} = A_0[\mathrm{i} + c_1 \tanh(\overline{\xi})] \exp\left[\mathrm{i}(k_2(t^2 + c_2 t) + k_{03})\right],$$

$$|q_{53}|^2 = A_0^2[1 + c_1^2 \tanh^2(\overline{\xi})], \tag{35}$$

where

$$\alpha_1 = -\frac{1}{4}\frac{k_{2,z}}{k_2^2}, \quad \Gamma = \frac{-A_0 k_{2,z} + 2A_{0,z} k_2}{2A_0 k_2},$$

$$\alpha_2 = +\frac{1}{2}\frac{c_2^2 k_{2,z}}{c_1^2 A_0^2}, \quad \overline{\xi} = c_2 k_2 (t + \frac{c_3 c_1 + c_2}{2c_1}) + c_5,$$

$$k_{03} = \frac{2c_1^2 c_2^2 + c_1^2 c_3^2 + 2c_2^2}{4c_1^2} k_2 + c_4. \tag{36}$$

In the above results, $\alpha_3 = \alpha_4 = \alpha_5 = 0$.

Remark 6. From above expressions, we can derive

$$k_2 = \frac{1}{\int 4\alpha_1 \mathrm{d}z + c_0}, \quad \Gamma = \frac{\alpha_1' \alpha_2 - \alpha_1 \alpha_2'}{2\alpha_1 \alpha_2} - 2\alpha_1 k_2. \tag{37}$$

Therefore, the solutions q_{51}, q_{52} and q_{53} are changed into the following forms:

$$q_{51} = \mathrm{i}c_1 \sqrt{\frac{2\alpha_1}{\alpha_2}} k_2 \mathrm{sech}(\xi) \exp[\mathrm{i}(k_2(t^2 + c_2 t) + k_{01})], \tag{38}$$

$$q_{52} = c_1 \sqrt{-\frac{2\alpha_1}{\alpha_2}} k_2 \tanh(\xi) \exp[\mathrm{i}(k_2(t^2 + c_2 t) + k_{02})], \tag{39}$$

$$q_{53} = \frac{c_2}{c_1} \sqrt{-\frac{2\alpha_1}{\alpha_2}} k_2 [\mathrm{i} + \mathrm{c}_1 \tanh(\bar{\xi})] \cdot \exp[\mathrm{i}(k_2(t^2 + c_2 t) + k_{03})]. \tag{40}$$

It is easy to see that when $c_2 = 0$, Theorems 1 and 2 in [26] can be reproduced by our results (38) and (39). At the same time, the solutions (51) and (52) obtained in [22] can also be reproduced by (38) and (39). But to our knowledge, the other solutions have not been reported earlier.

In the following, we turn our attention to discuss some special dispersion-managed soliton. Because the fundamental set of dispersion-managed solutions can be expressed in trigonometric and hyperbolic functions [26], we take the parameters α_1, α_2, Γ as the forms

$$\alpha_1 = 1 + \lambda \sin^2(z), \quad \alpha_2 = \theta,$$
$$k_2 = \frac{1}{(4 - 2\lambda)z - \lambda \sin(2z) + c_0}, \tag{41}$$
$$\Gamma = \frac{\lambda}{2} \frac{\sin(2z)}{1 + \lambda \sin^2(z)} - \frac{2(1 + \lambda \sin^2(z))}{(4 - 2\lambda)z - \lambda \sin(2z) + c_0},$$

where λ, θ are constants.

Figure 5a represents the dispersion-managed (DM) bright solitary wave propagation (38) and Fig. 5b represents two DM bright solitary waves interaction (38) under periodic functions conditions (41). As shown in Fig. 5c and 5d, under the parameter conditions (41), the solutions (39) and (40) present dispersion-managed dark solitary waves.

3. Summary and Discussion

In this paper, we have investigated the exact analytical solutions to the inhomogeneous higher-order nonlinear Schrödinger (IHNLS) equation including not only the group velocity dispersion and self-phase-modulation, but also various higher-order effects, such as the third-order dispersion, self-steepening and self-frequency shift. With the help of symbolic computation, a broad class of analytical solutions of the IHNLS equation is presented by the generalized projective Riccati equation method, which includes bright solitary wave solutions, dark solitary wave solutions, combined bright and dark solitary wave solutions, and dispersion-managed solitary wave solutions. From our results, many previously known results of the IHNLS equation can be recovered by means of some suitable selections of the arbitrary functions and arbitrary constants. Furthermore, from the soliton management concept, the main soliton-like character of the exact analytical solutions is discussed and simulated by computer under different parameters conditions.

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